

Conformal Black Hole Solutions of Axi-Dilaton Gravity in D-dimensions

H. Cebeci , T. Dereli

Department of Physics, Middle East Technical University
06531 Ankara, Turkey

Static, spherically symmetric solutions of axi-dilaton gravity in D dimensions is given in the Brans-Dicke frame for arbitrary values of the Brans-Dicke constant ω and an axion-dilaton coupling parameter k . The mass and the dilaton and axion charges are determined and a BPS bound is derived. There exists a one parameter family of black hole solutions in the scale invariant limit.

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I. INTRODUCTION

It is an exciting conjecture that all superstring models belong to a hypothetical eleven dimensional M-theory that would accommodate their apparent dualities. M-theory as a classical theory may be considered in a low-energy limit where only the low-lying massless excitation modes contribute to an effective field theory. As such it would be the same as the eleven dimensional simple supergravity theory. A subsequent Kaluza-Klein reduction would bring it to a ten dimensional theory related with the type IIA string model whose gravitational sector consists of the space-time metric tensor g , dilaton scalar ϕ and the axion potential $(p+1)$ -form A that would minimally couple to p -branes. We call such an effective gravitational field theory an axi-dilaton gravity in D -dimensions and consider in the following its static, spherically symmetric solutions for $p = D - 4$.

The study of black hole solutions of higher dimensional gravity theories started in 1963 with the generalisation of Schwarzschild and Reissner-Nordström solutions to $D > 4$ dimensions by Tangherlini [1]. These solutions were later put in a wider context by Myers and Perry [2], while Gibbons and Maeda [3] emphasised the relevance of dilaton scalars for the interpretation of such solutions. They provided a wide range of static, spherically symmetric solutions of the coupled Einstein-antisymmetric tensor-massless scalar field equations (see also [4], [5]). On the other hand it is a well-known fact that scalar-tensor theory of Brans-Dicke [6] may be re-written in terms of a conformally re-scaled metric as the coupled Einstein-massless scalar field theory [7], [8], [9]. For a particular value of the Brans-Dicke coupling parameter, namely for $\omega = -\frac{3}{2}$ in 4-dimensions, theory becomes locally scale invariant and called the Einstein-conformal scalar field theory. We had shown in a previous work that the conformal re-scaling properties of the Brans-Dicke theory can be conveniently exhibited using the non-Riemannian reformulation involving space-time torsion expressed in terms of the gradient of the scalar field [10]. Brans-Dicke theory has also been generalised to D -dimensions [11] and the black hole solutions of the Brans-Dicke-Maxwell field equations were given [12], [13].

In a remarkable paper Bekenstein [14] has found two classes of static, spherically symmetric solutions of the Einstein-conformal scalar field equations, and argued [15] that one particular class describes black hole solutions with scalar hair. His arguments were later repeated in $D > 4$ dimensions [16]. It is essential here to note that such a subclass of conformal black hole solutions cannot be reached by the assumptions of Ref. [3]. In this paper we consider axi-dilaton gravity in D -dimensions ($p = D - 4$) in the Brans-Dicke frame and give its static, spherically symmetric solutions for arbitrary values of two coupling parameters ω and k . A one-parameter family of conformal black hole solutions are obtained for $\omega = -\frac{D-1}{D-2}$ and $k = -\frac{D-4}{D-2}$.

II. AXI-DILATON GRAVITY IN D -DIMENSIONS

The dynamics of the axi-dilaton gravity will be determined by a variational principle from the action $I[e, \omega, \phi, A] = \int \mathcal{L}$ where the Lagrangian density D -form is taken in the Brans-Dicke frame as

$$\mathcal{L} = \frac{\phi}{2} R_{ab} \wedge *(e^a \wedge e^b) - \frac{\omega}{2\phi} d\phi \wedge *d\phi - \frac{\phi^k}{2} H \wedge *H . \quad (2.1)$$

Here the basic gravitational field variables are the co-frame 1-forms e^a , in terms of which the space-time metric $g = \eta_{ab} e^a \otimes e^b$ where $\eta_{ab} = \text{diag}(-+++ \dots)$. The Hodge $*$ map is defined so that the oriented volume form

$*1 = e^0 \wedge e^1 \wedge \dots \wedge e^n$. The metric compatible, torsion-free connection 1-forms ω^a_b are obtained from the Cartan structure equations

$$de^a + \omega^a_b \wedge e^b = 0, \quad (2.2)$$

and the corresponding curvature 2-forms

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b. \quad (2.3)$$

ϕ is the dilaton 0-form and H is a $(p+2)$ -form field that is derived from the $(p+1)$ -form axion potential A such that $H = dA$. ω and k are real parameters.

The field equations obtained from this action are

$$-\frac{\phi}{2} R^{bc} \wedge *(e_a \wedge e_b \wedge e_c) = \frac{\omega}{\phi} \tau_a[\phi] + \phi^k \tau_a[H] + D(\iota_a * d\phi) \quad , \quad (2.4)$$

$$\tilde{k} d * d\phi = \frac{\alpha}{2} \phi^k H \wedge *H \quad , \quad (2.5)$$

$$d(\phi^k * H) = 0 \quad , \quad dH = 0, \quad (2.6)$$

where the dilaton and axion stress-energy $(D-1)$ -forms are given by

$$\tau_a[\phi] = \frac{1}{2} (\iota_a d\phi \wedge *d\phi + d\phi \wedge \iota_a *d\phi) \quad (2.7)$$

and

$$\tau_a[H] = \frac{1}{2} (\iota_a H \wedge *H + (-1)^{p-1} H \wedge \iota_a *H), \quad (2.8)$$

respectively. We set $\alpha = \frac{2p-(n-3)}{n-1} + k$ and $\tilde{k} = \frac{n}{n-1} + \omega$.

The same action may be rewritten in terms of the $(D-p-2)$ -form field

$$G \equiv \phi^k * H \quad (2.9)$$

that is dual to the axion $(p+2)$ -form field H . We have in terms of G ,

$$\mathcal{L} = \frac{\phi}{2} R_{ab} \wedge *(e^a \wedge e^b) - \frac{\omega}{2\phi} d\phi \wedge *d\phi + \frac{\phi^{-k}}{2} G \wedge *G. \quad (2.10)$$

Hence given any solution $\{g, \phi, H\}$ of the field equations derived from (1), we may write down a dual solution $\{g, \phi, G\}$ to the field equations derived from (10). This notion of duality generalises the usual electric-magnetic duality in $D=4$ source-free electromagnetism.

Finally we wish to point out that the passage to the Einstein frame is achieved by the following conformal re-scaling of the field variables:

$$\begin{aligned} \tilde{g} &= \phi^{\frac{2}{n-1}} g \\ \tilde{\phi} &= \tilde{k}^{1/2} \ln \phi \\ \tilde{H} &= H. \end{aligned} \quad (2.11)$$

The resulting Lagrangian density D-form will be

$$\mathcal{L} = \frac{1}{2} \tilde{R}_{ab} \wedge \tilde{*} (\tilde{e}^a \wedge \tilde{e}^b) - \frac{1}{2} d\tilde{\phi} \wedge \tilde{*} d\tilde{\phi} - \frac{1}{2} \exp\left(\frac{\alpha}{\tilde{k}^{1/2}} \tilde{\phi}\right) \tilde{H} \wedge \tilde{*} \tilde{H}. \quad (2.12)$$

Given the above information, it is not difficult to compare solutions obtained in the Brans-Dicke frame with those given in the Einstein frame.

III. STATIC, SPHERICALLY SYMMETRIC SOLUTIONS

We will be giving below the most general static, spherically symmetric $p = (D - 4)$ brane solution to the field equations (2.4)-(2.6). This family of solutions generalises the usual magnetically charged Reissner-Nordström black hole solution in $D = 4$ to higher dimensions in a natural way. To this end we start with the ansatz

$$g = -f^2(r)dt \otimes dt + h^2(r)dr \otimes dr + R^2(r)d\Omega_{n-1} \quad (3.1)$$

for the metric tensor ($D = n + 1$),

$$\phi = \phi(r)$$

for the dilaton 0-form and

$$H = g(r)e^1 \wedge e^2 \wedge e^3 \dots \wedge e^{n-1}$$

for the axion field ($D - 2$)-form. We set $e^0 = f(r)dt$ and $e^n = h(r)dr$. Then the Einstein field equations reduce to the following set of ordinary coupled differential equations: (' denotes partial derivative with respect to r)

$$\begin{aligned} \phi \left[\frac{(n-2)(n-1)h}{2R^2} \left(1 - \left(\frac{R'}{h} \right)^2 \right) - \frac{(n-1)}{R} \left(\frac{R'}{h} \right)' \right] \\ = \frac{\omega}{2\phi} \left(\frac{\phi'^2}{h} \right) + \frac{\phi^k}{2} g^2 h + \left(\frac{\phi'}{h} \right)' + (n-1) \frac{\phi' R'}{hR}, \end{aligned} \quad (3.2)$$

$$\begin{aligned} \phi \left[\frac{(n-2)f'R'}{hR} + \left(\frac{f'}{h} \right)' + \frac{(n-2)f}{R} \left(\frac{R'}{h} \right)' - \frac{(n-3)(n-2)fh}{2R^2} \left(1 - \left(\frac{R'}{h} \right)^2 \right) \right] \\ = -\frac{\omega f}{2\phi h} \phi'^2 + \frac{\phi^k g^2 f h}{2} - \left(\frac{\phi' f}{h} \right)' - (n-2) \frac{\phi' f}{hR}, \end{aligned} \quad (3.3)$$

$$\begin{aligned} \phi \left[\frac{(n-1)(n-2)f}{2R^2} \left(1 - \left(\frac{R'}{h} \right)^2 \right) - (n-1) \frac{f'R'}{h^2 R} \right] \\ = -\frac{\omega f \phi'^2}{2\phi h^2} + \frac{g^2 f \phi^k}{2} + \frac{f'\phi'}{h^2} + (n-1) \frac{fR'\phi'}{h^2 R}; \end{aligned} \quad (3.4)$$

while the dilaton field equation becomes

$$\tilde{k}(\phi' \frac{f}{h} R^{n-1})' = \frac{\alpha}{2} \phi^k g^2 f h R^{n-1} \quad (3.5)$$

and the axion field equation reads

$$(gR^{n-1})' = 0. \quad (3.6)$$

Solutions to the above field equations can be written as:

$$\begin{aligned} R(r) &= r \left(1 - \left(\frac{C_1}{r} \right)^{n-2} \right)^{\alpha_3} \\ f(r) &= \left(1 - \left(\frac{C_2}{r} \right)^{n-2} \right)^{\alpha_4} \left(1 - \left(\frac{C_1}{r} \right)^{n-2} \right)^{\alpha_5} \\ h(r) &= \left(1 - \left(\frac{C_2}{r} \right)^{n-2} \right)^{\alpha_2} \left(1 - \left(\frac{C_1}{r} \right)^{n-2} \right)^{\alpha_1} \end{aligned} \quad (3.7)$$

$$\phi = \left(1 - \left(\frac{C_1}{r} \right)^{n-2} \right)^{\frac{2\gamma}{\alpha}}$$

$$g(r) = \frac{Q}{R^{n-1}}$$

where C_1 and C_2 are two independent integration constants and the third integration constant

$$Q^2 = \frac{4\tilde{k}\gamma(C_1C_2)^{n-2}(n-2)^2}{\alpha^2}.$$

The exponents are

$$\alpha_1 = \gamma\left(\frac{1}{(n-2)} - \frac{2}{(n-1)\alpha}\right) - \frac{1}{2}, \quad \alpha_2 = -\frac{1}{2},$$

$$\alpha_3 = \gamma\left(\frac{1}{(n-2)} - \frac{2}{(n-1)\alpha}\right),$$

$$\alpha_4 = \frac{1}{2}, \quad \alpha_5 = -\gamma\left(1 + \frac{2}{(n-1)\alpha}\right) + \frac{1}{2},$$

with

$$\gamma = \frac{\alpha^2(n-1)}{4\tilde{k}(n-2) + \alpha^2(n-1)}.$$

Some special cases deserve attention:

i) For $Q = 0$ and $\phi = \text{const.}$, we obtain the Tangherlini solution [1] which is the generalisation of the Schwarzschild solution in $D = n + 1$ dimensions,

$$g = -\left(1 - \frac{2M}{r^{n-2}}\right) dt^2 + \left(1 - \frac{2M}{r^{n-2}}\right)^{-1} dr^2 + r^2 d\Omega_{n-1} \quad (3.8)$$

ii) For $k = 0$ and $\phi = \text{const.}$, we obtain the $D = n + 1$ -dimensional generalisation of the Reissner-Nordström metric

$$g = -\left(1 + \frac{Q^2}{(n-1)(n-2)r^{2(n-2)}} - \frac{2M}{r^{n-2}}\right) dt^2 + \left(1 + \frac{Q^2}{(n-1)(n-2)r^{2(n-2)}} - \frac{2M}{r^{n-2}}\right)^{-1} dr^2 + r^2 d\Omega_{n-1}. \quad (3.9)$$

The electric dual of this solution was also given by Tangherlini.

iii) For $Q = 0$ we obtain solutions that generalise the Janis-Newman-Winicour solutions of the Einstein-massless scalar field equations to D dimensions [4]:

$$\begin{aligned} R(r) &= rh(r) \\ f(r) &= \left(\frac{r^{n-2}-r_0^{n-2}}{r^{n-2}+r_0^{n-2}}\right)^{\beta_1-\beta_2} \\ h(r) &= \left(1 - \left(\frac{r_0}{r}\right)^{2(n-2)}\right)^{1/(n-2)} \left(\frac{r^{n-2}-r_0^{n-2}}{r^{n-2}+r_0^{n-2}}\right)^{-\beta_1/(n-2)-\beta_2} \\ \phi(r) &= \left(\frac{r^{n-2}-r_0^{n-2}}{r^{n-2}+r_0^{n-2}}\right)^{\beta_2} \end{aligned} \quad (3.10)$$

where in order to ease comparison we use the parametrisation

$$\beta_2 = \sqrt{\frac{4(n-1)}{(n-2)\tilde{k}}(4-\beta_1^2)}$$

and β_1 satisfies $4(n-2)r_0^{n-2}\beta_1 = C$, r_0 and C being integration constants.

A consideration of the asymptotic behaviour of the fields in Brans-Dicke frame will allow us to determine a relationship satisfied by the mass, dilaton charge and magnetic charge Q . The mass of the black hole is defined to be

$$2M \equiv \lim_{r \rightarrow \infty} r^{n-2}(1 - f^2) = (C_2)^{n-2} + (\tilde{\gamma} - 2\gamma)(C_1)^{n-2} \quad (3.11)$$

where $\tilde{\gamma} = 1 - \frac{4\gamma}{(n-1)\alpha}$. The scalar charge

$$\Sigma \equiv \lim_{r \rightarrow \infty} r^{n-1} \frac{\phi'}{\phi} = 2(n-2) \frac{\gamma}{\alpha} (C_1)^{n-2}. \quad (3.12)$$

Finally the magnetic charge can be found from

$$Q \equiv \lim_{r \rightarrow \infty} r^{n-1} g = Q. \quad (3.13)$$

Therefore, by eliminating the integration constants C_1 and C_2 above we can find the following relationship between these 3 physical parameters:

$$Q^2 = \frac{2(n-2)\Sigma}{\alpha} \tilde{k} \left[(2\gamma - \tilde{\gamma}) \frac{\alpha\Sigma}{2(n-2)\gamma} + 2M \right]. \quad (3.14)$$

From this relationship, since Σ is a real parameter, the BPS bound respected by the mass and charge of a black hole follows:

$$M \geq \frac{1}{2(n-2)} \sqrt{\frac{4(n-2)\tilde{k} + 4\alpha - \alpha^2(n-1)}{(n-1)\tilde{k}}} |Q| \quad (3.15)$$

provided

$$\alpha^2(n-1) - 4\alpha \leq 4(n-2)\tilde{k}. \quad (3.16)$$

IV. CONCLUSION

A conformally scale invariant theory is obtained for the parameter values $\omega = -\frac{n}{(n-1)}$ and $k = -\frac{(n-3)}{(n-1)}$. A class of static, spherically symmetric solutions to the conformally scale invariant theory may be reached from the solutions (3.7) above by taking the limit $\alpha \rightarrow 0$ and $\tilde{k} \rightarrow 0$ with the ratio $\frac{2\gamma}{\alpha}$ kept fixed:

$$\begin{aligned} R(r) &= r \left(1 - \left(\frac{C_1}{r} \right)^{n-2} \right)^{-\frac{\beta}{(n-1)}} \\ f(r) &= \left(1 - \left(\frac{C_2}{r} \right)^{n-2} \right)^{1/2} \left(1 - \left(\frac{C_1}{r} \right)^{n-2} \right)^{1/2 - \frac{\beta}{(n-1)}} \\ h(r) &= \left(1 - \left(\frac{C_2}{r} \right)^{n-2} \right)^{-1/2} \left(1 - \left(\frac{C_1}{r} \right)^{n-2} \right)^{-1/2 - \frac{\beta}{(n-1)}} \\ \phi &= \left(1 - \left(\frac{C_1}{r} \right)^{n-2} \right)^\beta \\ g(r) &= \frac{Q}{R^{n-1}} \end{aligned} \quad (4.1)$$

where C_1 and C_2 are constants and β and Q should satisfy

$$2\beta(n-2)^2(C_1 C_2)^{n-2} = Q^2.$$

The special case of parameter values $Q = 0$ and $C_2 = 0$ in $D = 4$ dimensions brings (4.1) to Bekenstein's Einstein-conformal scalar solution [14]. The fact that this solution describes black holes was later clarified by Bekenstein [15]. His argument is based on the observation that the scalar particles being postulated to follow geodesic world-lines in Brans-Dicke theory [17] pre-supposes a particular type of scalar field coupling to matter. On the other hand by assuming a different type of matter couplings one can show that neutral test particles would follow conformal world-lines as argued by Gürsey [18] and Dirac [19]. This assumption implies in particular that solution (4.1) describes a black hole with finite scalar charge. In a recent paper it was shown that the conformal world-lines are nothing but autoparallel curves in the non-Riemannian re-formulation of the Brans-Dicke theory [20]. A further re-evaluation of the locally scale invariant solutions above from a non-Riemannian point of view will be taken up in a separate study.

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